

# BPS states in Matrix Strings

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Matrix string theory (or more generally U-Duality) requires Super Yang-Mills theory to reflect a stringy degeneracy of BPS short multiplets. These are found as supersymmetric states in the Yang-Mills carrying (fractionated) momentum, or in some cases, instanton number. Their energies also agree with those expected from M(atrix) theory. A nice parallel also emerges in the relevant cases, between momentum and instanton number, (both integral as well as fractional) providing evidence for a recent conjecture relating the two.

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## 1. Introduction

In its most recent phase of development, string theory seems to be emerging from the confining cocoon of Riemann surfaces. While it is not clear what it will finally metamorphose into, some features may be dimly discerned. One of them is the derivative nature of spacetime, perhaps embedded in a non-commutative space. Another is a milder exponential growth of the number of states as compared to the free string degeneracy. This latter feature, reflected in the counting of black hole microstates, is due to the fact that it is the BPS states that survive quantum corrections. The recent provocative proposal of [1] tries to minimally incorporate these ideas.

But it was clear even earlier, with the identification of D-branes with various RR solitons, that the properties and degeneracies of BPS states must be reflected in Super Yang-Mills theory. Through this process we have been uncovering rather remarkable properties of Yang-Mills theory. In this paper we'll see a little of that in the fractionation of momentum in field theory and its relation to instanton number.

Let us outline the problem: The conjecture made in [1] essentially implies that the relevant degrees of freedom for M-theory on a transverse  $T^d$ , in the infinite momentum frame, are those of a large N Yang-Mills theory on  $T^d \times R$  with the equivalent of four dimensional  $N = 4$  supersymmetry [2]. We shall have occasion to deal with various tori, but let us, for now, consider M-theory on  $T^2$ . The relation with type II string theory on a circle implies that the  $(2 + 1)$  dimensional Yang-Mills theory is to possess a whole stringy tower of states. Given the relation between couplings, the perturbative string spectrum is expected to show up only in the strong coupling limit of the field theory. Some arguments have recently been advanced [3][4] that the non-trivial conformal field theory in this limit might indeed have this property. However we certainly expect the BPS states of the string theory to show up in the field theory for any value of the coupling, in particular, a semi-classical analysis is accurate given the amount of supersymmetry. These states could either be “ultra- short” or “short” depending on whether they break 1/2 or 3/4 of the supersymmetry. Given the important role that this subset of the states play, it seems worthwhile to isolate them in the field theory.

The ultra-short multiplets are non-degenerate ( i.e. only 1 multiplet with  $16^2$  degrees of freedom): in the perturbative string spectrum they are states with both left and right moving oscillators unexcited. As perturbative states they have purely winding or momentum modes with respective masses, (we'll use string units  $\alpha' = 1$ ),

$$M^2 = (nR)^2; \quad M^2 = \left(\frac{m}{R}\right)^2. \quad (1.1)$$

Here  $m$  and  $n$  are the momentum and winding modes on the circle of radius  $R$ . On a more general torus, these multiplets could have both winding and momentum but in independent directions, thus for example, having a mass

$$M^2 = (n_1 R_1)^2 + (\frac{m_2}{R_2})^2. \quad (1.2)$$

U-duality relates this, for instance, to an  $(n_1, m_2)$  string and the existence of a unique bound state of the latter [5] was a check of U-duality [6]. These states correspond to vacua of the  $SU(N)$  sector of the  $U(N)$  Yang-Mills theory. Ultra-short multiplets were examined in the M(atrix) theory context in [7], where their energy was found to agree with expectations. Note that, being vacua of  $SU(N)$ , the energy is purely a  $U(1)$  contribution.

The short multiplets on the other hand have an enormous degeneracy. For instance, they appear in the perturbative spectrum on a circle with mass

$$M^2 = (\frac{m}{R})^2 + (nR)^2 + 2N_L = (\frac{|m|}{R} + |n|R)^2. \quad (1.3)$$

Here  $N_L$ , the left oscillator level, is given by the level matching condition to be  $N_L = |mn|$  and  $N_R = 0$ . The number of states is therefore given by the number of partitions  $d(mn)$  of level  $mn$  among the physical 8 bosonic and 8 fermionic oscillators (after taking into account the degeneracy ( $16^2$ ) of the ground state).

$$\sum d(k)q^k = 16^2 \prod_{m=1}^{\infty} (\frac{1+q^m}{1-q^m})^8. \quad (1.4)$$

The modest problem that we will tackle here, is to find these states <sup>1</sup> in the Yang-Mills and check that the energies and degeneracies match. This, it will be seen, requires the field theory to fractionate its momentum. (A phenomenon familiar in the black hole state counting context [8][9][10][11]. See also the related recent discussion about long matrix strings [12][13][3][4] etc.) The field theory accomplishes this in an interesting fashion, very analogous to the fractionation of instanton number, to which also we'll relate this phenomenon.

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<sup>1</sup> Note that an important difference from the previous case, (due to the breaking of further supersymmetry) is that these are supersymmetric BPS states in the  $SU(N)$  Yang-Mills and not vacua. Consequently, the energy will also receive a non-abelian contribution, but still classically computable.

Actually, the fact that the degeneracies should agree is really a consequence of U-duality. And this exercise may also be viewed as a check of the latter. We'll see, in what follows, the relation to some earlier and different checks of U-duality [14][15]. The energies, however, computed in the field theory for any finite  $N$ , will agree with those in M-theory only in the infinite momentum frame with longitudinal momentum proportional to  $N$  as in [1].

The next section discusses the rather special case of M(atrrix) theory on  $T^4$ . Here the short multiplets will appear in the Yang-Mills as supersymmetric states possessing fractional  $(SU(N)/Z_N)$  instanton number. It will, in this sense, be a generalisation of the system studied by [14][15] and a useful preparatory case. In Section 3 we will study the more generic torus where the field theory fractionates its momentum. The short multiplets of string theory are directly visible in this case. Section 4 returns to  $T^4$  and discusses the parallel between instanton number and momentum. We end with some discussion and conclusions. An appendix exhibits some relevant field configurations with fractional  $SU(N)$  momentum or instanton number.

## 2. Magnetic Fluxes and Fractional Instanton Number

Let's start with M(atrrix) theory on  $T^4$  – it will soon be clear why we choose to do so. The  $N$  0-branes, in this case, lead a T-dualised existence as 4-branes [1][2]. Thus we will examine  $(4+1)$  dimensional  $U(N)$  Super-Yang-Mills theory on  $T^4 \times R$  for a stringy degeneracy of BPS states. The sides of the torus will be assumed to be of length  $a_i, i = 1, 2, 3, 4$ . (We will use roman letters for spatial directions only.) It was 't Hooft's observation [16] that one could have different topological sectors corresponding to the gauge fields being periodic upto a gauge transformation. These “twists” are labelled by, in our case, six integers  $n_{ij} = -n_{ji}$  defined modulo  $N$  and may be thought of as discrete magnetic fluxes  $F_{ij}$ . Since  $U(N) = (SU(N) \times U(1))/Z_N$ , the twist  $e^{2\pi i n_{ij}/N}$  in the  $SU(N)$  is accompanied by one of  $e^{-2\pi i n_{ij}/N}$  in the  $U(1)$  [17].

We'll consider the  $SU(N)$  and  $U(1)$  sectors individually, starting with the former. We restrict ourselves to backgrounds with static gauge fields (no scalars/fermions). Therefore we have for the  $SU(N)$  Hamiltonian a bound familiar from Euclidean four dimensions:

$$H^{SU(N)} = \frac{1}{4g_{YM5}^2} Tr \int d^4x F_{ij}^{SU(N)} F_{ij}^{SU(N)} \geq \left| \frac{1}{4g_{YM5}^2} Tr \int d^4x F_{ij}^{SU(N)} \tilde{F}_{ij}^{SU(N)} \right|. \quad (2.1)$$

It was shown in [16] that, in the presence of twisted boundary conditions, that the  $SU(N)$  instanton (more properly soliton) number can be fractional, in fact,

$$\frac{1}{16\pi^2} \text{Tr} \int d^4x F_{ij}^{SU(N)} \tilde{F}_{ij}^{SU(N)} = \nu - \frac{\kappa}{N}. \quad (2.2)$$

where  $\kappa = \frac{1}{4}n_{ij}\tilde{n}_{ij} = n_{12}n_{34} + n_{13}n_{42} + n_{14}n_{23}$ . The bound on the energy is saturated for (anti) self dual fields  $F_{ij}^{SU(N)} = \pm \tilde{F}_{ij}^{SU(N)}$ . (See appendix for some explicit configurations.) In the supersymmetric theory at hand, it is by now a familiar fact that, these are the BPS configurations since the supersymmetry variation of the  $SU(N)$  gaugino is given by

$$\delta\chi^{SU(N)} = \Gamma^{ij} F_{ij}^{SU(N)} \epsilon. \quad (2.3)$$

A (anti) self-dual field implies that for half of the  $\epsilon$ 's (those of the appropriate chirality), the variation vanishes.

Let us remain in a sector where the integer component  $\nu$  is zero. Moreover, for simplicity, we'll also take only  $n_{12}, n_{34}$  non-zero. (This is also the (422) system studied in [17].) We then have a BPS state with energy given by

$$H^{SU(N)} = \frac{4\pi^2}{g_{YM5}^2 N} |n_{12}n_{34}| \quad (2.4)$$

There is a  $U(1)$  contribution to the energy too. This is a result of the identification of the twists of the  $U(1)$  and the  $SU(N)$ . We have constant  $U(1)$  magnetic fluxes

$$F_{ij}^{U(1)} a_i a_j = \frac{2\pi n_{ij}}{N} \quad (2.5)$$

with no sum over indices. (These  $U(1)$  fluxes give an opposite fractional contribution to the total  $U(N)$  instanton number so that the net number is zero [17].) This does not break any further supersymmetries since

$$\delta\chi^{U(1)} = \Gamma^{ij} F_{ij}^{U(1)} \epsilon + \epsilon', \quad (2.6)$$

and one can choose  $\epsilon'$  to cancel the  $\epsilon$  variation. The  $U(1)$  part of the energy is therefore

$$H^{U(1)} = \frac{1}{2g_{YM5}^2} \int d^4x ((F_{12}^{U(1)})^2 + (F_{34}^{U(1)})^2) = \frac{2\pi^2}{g_{YM5}^2 N} \left( \frac{a_3 a_4}{a_1 a_2} n_{12}^2 + \frac{a_1 a_2}{a_3 a_4} n_{34}^2 \right). \quad (2.7)$$

The total energy is now

$$H^{U(N)} = H^{U(1)} + H^{SU(N)} = \frac{2\pi^2 a_1 a_2 a_3 a_4}{g_{YM5}^2 N} \left( \frac{|n_{12}|}{a_1 a_2} + \frac{|n_{34}|}{a_3 a_4} \right)^2. \quad (2.8)$$

What about the degeneracy of a state with this energy? The moduli space of instantons in  $SU(N)/Z_N$  gauge theory with instanton number  $\nu$  is believed to be  $(T^4)^{N\nu}/S(N\nu)$  [15]. This is presumably true for both fractional and integral instanton number  $\nu$  since one can separate a configuration with integral instanton number into clusters with fractional charge. Quantising the collective coordinates and looking at the ground states of the corresponding supersymmetric quantum mechanics, one finds the degeneracy  $d(N\nu)$  to be given by the dimension of the cohomology of the above orbifold. Its generating function has been calculated in [18] to be precisely (1.4). (The factor of  $16^2$  comes from the  $U(1)$  sector – it was present in the case of short multiplets as well.) For a fractional instanton number  $\nu = n_{12}n_{34}/N$ , the degeneracy is  $d(n_{12}n_{34})$ .

Now for some comparisons. The configuration we have considered is in the language of 4-branes, a collection of  $N$  4-branes together with  $n_{12}, n_{34}$  2-branes wrapped on directions (34) and (12) respectively.<sup>2</sup> This follows from the fact that fluxes  $F_{12}$  and  $F_{34}$  act as sources of 2-brane charge in the 4-brane world volume. This is a short multiplet configuration which is U-dual to the perturbative state with momentum  $n_{12}$  and winding  $n_{34}$ , both in, say, direction 2. (This may be seen on following this configuration through the dualities  $T_1ST_{1234}ST_1$ .  $T_i$  denotes T-duality in direction  $i$ . The 4-branes go over into winding  $N$  in the 1 direction.) It is satisfying that we saw precisely the degeneracy  $d(n_{12}n_{34})$  in the gauge theory that we expect for this short state in string theory.

This is closely related to the results of [14][15]. They consider in the system of  $N$  4-branes,  $m$  0-branes instead of the system of 2-branes that we had above. In the gauge theory, this corresponds to the sector with integral instanton number  $m$ . This is U-dual via the same duality chain in the previous paragraph to winding  $N$  and momentum  $m$  in direction 1. The degeneracy is thus  $d(Nm)$  which is also the degeneracy of states with integral instanton number  $m$  in the  $SU(N)$  theory. We can now easily see that it is possible to generalise to the case of arbitrary instanton number by combining 0-branes and 2-branes. The degeneracies are precisely those expected of a string state with winding and momentum simultaneously in directions 1 and 2.

In the M(atrix) theory picture, our gauge configuration is (the T-dual of) the configuration with  $n_{12}, n_{34}$  2 branes wrapped on directions (12) and (34) respectively, with  $N$  0-branes as always. This follows, again, from the identification of magnetic fluxes,  $F_{12}$  and

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<sup>2</sup> The system of 2-branes at angles has also recently been studied from the point of view of fractional instantons [19]

$F_{34}$  with 2-branes [7]. As we saw, having only fractional instanton charge means, in the language of 4-branes, that there are no 0-branes. Or equivalently, no longitudinal 5-branes in M(atrrix) theory.

We'll compare the energy (2.8) in gauge theory with that expected of this configuration of branes in M-theory. The mass of this bunch of wrapped 2-branes in M-theory is (written in terms of type II string parameters)

$$M = \frac{(|n_{12}|l_1l_2 + |n_{34}|l_3l_4)}{g_s}, \quad (2.9)$$

where  $l_i$  are the radii of the  $S^1$ 's. To relate to the gauge theory variables recall that (by T-duality [2][1])  $l_i = \frac{2\pi}{a_i}$ . We also need the identification of the (open) string coupling with the gauge coupling

$$\frac{1}{g_{YM5}^2} = (2\pi)^2 \frac{l_1l_2l_3l_4}{g_s}; \quad g_s = R_{11} \quad (2.10)$$

and the fact that the energy in the infinite momentum frame is given by

$$E = \frac{M^2}{2p_{11}}; \quad p_{11} = \frac{N}{R_{11}}. \quad (2.11)$$

Putting it all together gives precisely the energy of the state in gauge theory (2.8).

A few words should be said about ultrashort configurations. If we had instead turned on fluxes, say,  $n_{12}, n_{24}$ , then we would not have had any fractional instanton number. We would have had zero energy from the  $SU(N)$  part – in other words, a unique vacuum state. All the energy would have come from (2.7)(with index 3 replaced by 2). It is easy to check that the set of 2-branes this corresponds to, (wrapped on (34) and (13) directions) is U-dual to winding and momentum along different directions, an ultrashort state. The energy of this state in M-theory in the infinite momentum frame is also in agreement with (2.7),(compare with (1.2)).

### 3. Electric, Magnetic fluxes and Momentum

Thus far we have been on a special torus which admitted instantons and had switched on only the 't Hooft magnetic fluxes. We are now ready to consider a more general case. For definiteness, we'll mostly deal with M(atrrix) theory on  $T^2$ , though the generalisations will be evident. The relevant gauge theory is 2 + 1 dimensional with gauge group  $U(N)$ . We will also be turning on discrete electric fluxes. Physically, this is equivalent to the

presence of a Wilson line, which creates the electric flux, winding along one of the compact directions. The distinct topological sectors are labelled by integers  $q_i$  defined again modulo  $N$  corresponding to Electric fluxes  $F_{0i}$ .

As before, we first consider the  $SU(N)$  sector. We do not have an instanton bound, but rather the simpler (but less utilised)

$$H^{SU(N)} \geq |P_i| \equiv \left| \int d^2x T_{0i} \right| = \frac{1}{g_{YM3}^2} \text{Tr} \left| \int d^2x F_{0i}^{SU(N)} F_{ij}^{SU(N)} \right|. \quad (3.1)$$

The left hand side is simply the norm of the Non-abelian Poynting vector which measures the momentum in the field. Consider a topological sector with  $n_{12}, q_2 \neq 0$  and oscillator momentum  $m_1$  in direction 1. Then

$$H^{SU(N)} \geq \frac{1}{g_{YM3}^2} \text{Tr} \left| \int d^2x F_{0i}^{SU(N)} F_{ij}^{SU(N)} \right| = \frac{4\pi^2}{a_1} \left| m_1 - \frac{n_{12}q_2}{N} \right|. \quad (3.2)$$

It is no surprise to see the contribution  $\frac{4\pi^2 m_1}{a_1}$  to the momentum, (Note that  $a_i$  are the circumferences, not the radii) but the fractional term is not usual. However, it is there since one can write down explicit field configurations analogous to those with fractional instanton number (see appendix). And while one does not think of momentum as a topological quantum number, on a compact domain the quantisation makes it impossible for this number to change continuously on varying the field configuration. We thus see clearly in the field theory context how fractional momentum emerges. It is the momentum associated with the discrete fluxes that a  $SU(N)/Z_N$  gauge theory admits.

When is the bound on the energy satisfied? In the case we have been considering it occurs when

$$F_{02}^{SU(N)} = \pm F_{12}^{SU(N)} \equiv \hat{F}. \quad (3.3)$$

This may look unfamiliar but it is just the non-abelian generalisation of an electromagnetic plane wave travelling in direction 1. The electric and magnetic fields are orthogonal (in directions 2 and 3, say) and have equal norm. In our supersymmetric theory, it is nice fact that these are also BPS states. We see that (2.3) reduces to

$$\delta\chi^{SU(N)} \propto \Gamma^2(\Gamma^0 \pm \Gamma^1)\epsilon\hat{F}. \quad (3.4)$$

The ‘‘Dirac equation’’ structure immediately tells us that for half of the  $\epsilon$ ’s, the variation vanishes.



In analogy with the previous section, let us remain in the sector with  $m_1$  zero, i.e. with only fractional  $SU(N)$  momentum  $\frac{n_{12}q_2}{N}$ . (We must remind the reader that just as with instanton number, the nett  $U(N)$  momentum is always integral. But the spacing between levels is in units of  $\frac{1}{N}$ .) Then

$$H^{SU(N)} = \frac{4\pi^2 |n_{12}q_2|}{Na_1} \quad (3.5)$$

The  $U(1)$  contribution arises from the fact that the two sectors are correlated. As before we have

$$F_{12}^{U(1)} a_1 a_2 = \frac{2\pi n_{12}}{N}. \quad (3.6)$$

But now there is an Electric flux as well whose quantisation condition reads as

$$E_2^{U(1)} a_1 = \frac{2\pi q_2}{N}; \quad E_2^{U(1)} \equiv \frac{1}{g_{YM3}^2} F_{02}^{SU(N)} \quad (3.7)$$

Thus

$$H^{U(1)} = \frac{g_{YM3}^2}{2} \int d^2x (E_2^{U(1)})^2 + \frac{1}{2g_{YM3}^2} \int d^2x (F_{12}^{U(1)})^2 = \frac{2\pi^2 g_{YM3}^2}{N} q_2^2 \left(\frac{a_2}{a_1}\right) + \frac{2\pi^2 n_{12}^2}{g_{YM3}^2 N a_1 a_2}. \quad (3.8)$$

The total energy now reads as

$$H^{U(N)} = H^{U(1)} + H^{SU(N)} = \frac{2\pi^2 g_{YM3}^2}{a_1 a_2 N} \left(q_2 a_2 + \frac{n_{12}}{g_{YM3}^2}\right)^2 \quad (3.9)$$

Now for the degeneracy of these momentum states. Here, we do not yet have a complete argument. But it is plausible, given that momentum is quantised in units of  $\frac{1}{N}$ , that the degeneracy from the  $SU(N)$  sector will be given by the number of ways  $d(n_{12}q_2)$  of partitioning  $n_{12}q_2$  amongst 8 bosonic and fermionic modes. This is similar to the counting in [9].

We are now ready to make some comparisons. We have been essentially considering the world volume theory of  $N$  2-branes together with  $n_{12}$  0-branes and fundamental strings with winding  $q_2$ . This follows from the identification [5] of electric fluxes with winding of fundamental strings. Again, this is a short multiplet configuration which is U-dual to a perturbative string state with momentum  $n_{12}$  and winding  $q_2$ , both in direction 2. (As follows from the duality chain  $T_2ST_2$ . The 2-branes go over into strings with winding  $N$  in direction 1.) The degeneracy of this latter state is  $d(n_{12}q_2)$ , in accord with what we had in the Yang-Mills. The case with only integer momenta  $m_1$  in (3.2), is U-dual, via the same

chain, to  $m_1$  units of momentum together with winding  $N$  in direction 1, thus another short multiplet configuration. Now the degeneracy is  $d(Nm_1)$ , once again something that might be expected in the Yang-Mills given the fractional units of momentum. Again, the case of arbitrary momentum in (3.2) poses no problems, being the combination of the above two cases and has the right degeneracy. All this is a test of U-duality for M-theory on tori, which is independent from the one performed in [14][15], as we'll elaborate a bit in the next section.

On a general torus  $T^d$ , the considerations are very analogous. The  $U(N)$  Yang-Mills is the world-volume theory of  $N$   $d$  branes. One may have  $n_{12}$   $(d-2)$  branes wrapped on directions  $(3, \dots, d)$  as the source of magnetic flux and fundamental strings with winding  $q_2$  on direction 2 as that of electric flux. Integer momenta  $m_1$  may also be present. This would be dual (via  $T_{2\dots d}ST_{2\dots d}$  for even  $d$  and  $T_{1\dots d}ST_{1\dots d}$  for odd  $d$ ) to the short string multiplets of the previous paragraph. The counting works exactly as before. Only the mass formula has some modifications for the dimensionality. It is clear, for instance, that we will need to consider the  $(5+1)$  dimensional Field theory with configurations of both instantons and momenta to make contact with the states responsible for Black Hole entropy [20][21].

Coming to the M(atric) theory on  $T^2$  interpretation, our original gauge configuration is (the T-dual of) a collection of  $N$  0-branes,  $n_{12}$  transverse 2-branes wrapped on the  $T^2$  and  $q_2$  units of Kaluza-Klein momentum in direction 2. The case with momentum  $m_1$  corresponds to a longitudinal 2-brane wrapped on direction 1. We can compare the energy of this set-up with (3.9). The mass is given by (taking  $m_1 = 0$ )

$$M = \frac{|n_{12}|l_1l_2}{g_s} + \frac{|q_2|}{l_2}. \quad (3.10)$$

As before,  $l_i = \frac{2\pi}{a_i}$ . The gauge coupling is now given by

$$\frac{1}{g_{YM3}^2} = \frac{2\pi l_1 l_2}{g_s}. \quad (3.11)$$

Together with (2.11) we reproduce exactly (3.9).

There is another way of putting this so that we can directly see how gauge theory reproduces the perturbative string spectrum (1.3). Running the argument backwards, we could say that the Yang-Mills calculation (3.9) predicts, using the usual identifications (3.11) and (2.11), that the mass of a state in M-theory on  $T^2$  with  $n_{12}$  2-branes and KK momentum  $q_2$  is (3.10). Now M-theory on  $T^2$  is equivalent to string theory on  $S^1$ , taking

say, direction 1 to be the “eleventh” one. The  $n_{12}$  2-branes are, from this angle, fundamental strings with winding  $n_{12}$  on direction 2. The tension of this string is given by  $\frac{1}{2\pi\alpha'} = \frac{l_1}{2\pi g_s}$  – derived from the 2-brane wrapped on direction 1, (using the 2-brane tension derived in M(atrix) theory [1]). The KK momenta  $q_2$  in direction 2 remain what they are. We immediately see, with this identification of the tension, and  $l_2 \rightarrow R, n_{12} \rightarrow n, q_2 \rightarrow m$  that (3.10) coincides with (1.3).

#### 4. Instantons and Momenta

The development of the last two sections has been rather analogous. Instantons and momenta are both fractionated in units of  $\frac{1}{N}$ , compare equations (2.2) and (3.2). However the instantons appeared only in the case of  $T^4$ , while momenta are present in any dimension! So, let us focus, once more, on the case of (4+1) dimensional  $SU(N)$  Super Yang-Mills where they both occur, and examine the correspondence a bit better.

Let’s take integer  $SU(N)$  quantum numbers (though the conclusions would be the same for fractional charges) for momenta/instanton number. Thus the two physical situations are : 1.  $N$  4-branes with  $m$  0-branes, 2.  $N$  4-branes with momentum  $m$  in direction 1. There is a third configuration that they are both U-dual to, namely, the string state with winding  $N$  and momentum  $m$  in direction 1. These represent the three “conjugacy classes” for short multiplets under the U-duality group for String theory on  $T^4$ .<sup>3</sup> It is the first two that we have considered as BPS states in the Yang-Mills. We have seen that they have exactly the same degeneracy. In fact, the form of (2.2) and (3.2), suggest an interpretation of  $g_{YM5}^2$  as the size of a fifth circle, on the same footing as the  $a_i$ . Instantons (actually solitons) are then momenta in this direction. The fractionation of momenta might be used to motivate that of instanton number or vice versa. In fact, the identification of (integral) instanton number with momenta and the emergence of a new dimension is precisely the suggestion made recently in [22](see also [23]) . We can now give some more evidence for this conjecture. We have seen that there is a whole tower of BPS states which transform into each other under the postulated  $SL(5, Z)$  symmetry. Moreover, their energies also

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<sup>3</sup> Any two configurations equivalent upto T-duality (i.e. perturbatively equivalent) belong to the same “conjugacy class”. There are two such classes of short multiplets for string theory on  $T^d$ ,  $d < 4$  and three for  $d = 4$ . This is an equivalent reason as to why  $T^4$  is special.

reflect this invariance. If we redo the calculations (3.7)(3.8)(3.9) in the (4+1) dimensional case, we obtain for the energy of a state with  $n_{12}, q_2 \neq 0$

$$H^{U(N)} = \frac{2\pi^2 a_2 a_3 a_4 g_{YM5}^2}{a_1 N} \left( \frac{|q_2|}{a_3 a_4} + \frac{|n_{12}|}{g_{YM5}^2 a_2} \right)^2 \quad (4.1)$$

Comparing with (2.8), with the identification  $g_{YM5}^2 \equiv a_5$ , we immediately see the symmetry among the “five” directions. We also see that the Poynting vector and Instanton density are supposed to transform under the postulated  $SL(5, Z)$  symmetry of the theory. It might perhaps be possible to combine them into a symmetric form.

## 5. Discussion and Conclusions

The objective of this work was to find, purely within Yang-Mills theory a description of the BPS states of string theory. It is rather surprising that Yang-Mills theory seems to possess non-trivial characteristics which are demanded by string theory [24][3]. Here we have seen a hagedorn number of BPS states, carrying both momentum *and* winding quantum numbers and having the energies expected of the string spectrum. This was achieved by the field theory fractionating its momentum in a manner appropriate to long strings. This is done in a way so similar to that of instanton number that in the right circumstances, the interchange of instantons with momentum becomes a symmetry – one demanded once again by string theory. Of course, since this could be phrased in the context of M(atric) theory, we have equivalently made some new checks of U-duality and M(atric) theory. Our study have been for arbitrary  $N$ , but the energies coincide with that of M-theory/string theory only in the infinite momentum frame with the identification of the longitudinal momentum  $P_{11} = N/R_{11}$ . This is a general feature of toroidal compactifications of M(atric) theory (see remarks in [23]). Finally, the presence of a U-dual spectrum of states in the Yang-Mills is also a signature of Lorentz invariance.

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**Note Added:** While this manuscript was just being completed, we received the preprint [25] in which BPS states in the matrix theory were used to study black hole issues in 5 dimensions.

## Appendix

Following [16] and [17] we'll display some field configurations with fractional  $SU(N)$  instanton number and momentum. We'll see that they are very similar. More details can be had from these references.

For a (4+1) dimensional  $U(N)$  gauge theory with twists  $n_{12}, n_{34} \neq 0$ , the relevant fields are, for a self dual  $SU(N)$  configuration,

$$F_{12}^{SU(N)} = \frac{F}{N} \omega = F_{34}^{SU(N)} \quad (5.1)$$

where  $\omega = \text{diag}(\overbrace{k, \dots, k}^{l \text{ times}}, \overbrace{-l, \dots, -l}^{k \text{ times}})$  is the generator of the  $U(1)'$  in  $SU(l) \otimes SU(k) \otimes U(1)' \subset SU(N)$  with  $k + l = N$ .  $F$  is a constant determined by

$$F a_1 a_2 = \frac{2\pi n_{12}}{l}; \quad F a_3 a_4 = -\frac{2\pi n_{34}}{k}, \quad (5.2)$$

which also determines  $k, l$  in terms of the other parameters. The correlation of the twists between the  $SU(N)$  and  $U(1)$  parts imply that the  $U(1)$  field strengths are determined to be

$$F_{12}^{U(1)} = \frac{lF}{N} \mathbf{1}_{N \times N}; \quad F_{34}^{U(1)} = -\frac{kF}{N} \mathbf{1}_{N \times N}. \quad (5.3)$$

Note that the  $U(1)$  field strengths are not self-dual. We see that the  $SU(N)$  and  $U(1)$  instanton numbers are opposite in sign and equal in magnitude to  $\frac{n_{12}n_{34}}{N}$ .

The only modification in the case with electric fluxes is in some quantisations. If, as in Section 3.,  $n_{12}, q_2 \neq 0$ , then again

$$F_{12}^{SU(N)} = \frac{F'}{N} \omega = F_{02}^{SU(N)}. \quad (5.4)$$

with

$$F' a_1 a_2 = \frac{2\pi n_{12}}{l} \quad \frac{1}{g_{YM3}^2} F' a_1 = -\frac{2\pi q_2}{k}. \quad (5.5)$$

The  $U(1)$  fields are

$$F_{12}^{U(1)} = \frac{lF'}{N} \mathbf{1}_{N \times N}; \quad F_{34}^{U(1)} = -\frac{kF'}{N} \mathbf{1}_{N \times N}. \quad (5.6)$$

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